

Analytic Integrable Systems: Analytic Normalization and Embedding Flows

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Abstract

In this paper we mainly study the existence of analytic normalization and the normal form of finite dimensional complete analytic integrable dynamical systems. More details, we will prove that any complete analytic integrable diffeomorphism $F(x) = Bx + f(x)$ in $(\mathbb{C}^n, 0)$ with B having eigenvalues not modulus 1 and $f(x) = O(|x|^2)$ is locally analytically conjugate to its normal form. Meanwhile, we also prove that any complete analytic integrable differential system $\dot{x} = Ax + f(x)$ in $(\mathbb{C}^n, 0)$ with A having nonzero eigenvalues and $f(x) = O(|x|^2)$ is locally analytically conjugate to its normal form. Furthermore we will prove that any complete analytic integrable diffeomorphism defined on an analytic manifold can be embedded in a complete analytic integrable flow. We note that parts of our results are the improvement of Moser's one in *Comm. Pure Appl. Math.* 9(1956), 673–692 and of Poincaré's one in *Rendiconti del circolo matematico di Palermo* 5(1897), 193–239. These results also improve the ones in *J. Diff. Eqns.* 244(2008), 1080–1092 in the sense that the linear part of the systems can be nonhyperbolic, and the one in *Math. Res. Lett.* 9(2002), 217–228 in the way that our paper presents the concrete expression of the normal form in a restricted case.

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1. Introduction and statement of the main results

The study on the existence of analytic normalization for an analytic dynamical system to its normal form has a long history, which can be traced back to Poincaré and even earlier (see e.g. [28, 24, 5, 32, 31]). For analytic dynamical systems, if their analytically equivalent normal forms are known, it will be useful to study the dynamics of the original systems. It is well-known, see e.g. [28, 18, 19, 37, 32, 20] and the references therein, that the existence of analytic normalizations for analytic vector fields to their normal forms is also strongly related to the existence of analytic first integrals of analytic vector fields.

The aim of this paper is to settle the problems on the existence of analytic normalizations for analytic integrable diffeomorphisms to their normal forms and also for analytic integrable vector fields to their normal forms.

For a diffeomorphism $F(x)$ defined in $(\mathbb{C}^n, 0)$, a function $V(x)$ is an *analytic first integral* of $F(x)$ if it is analytic and satisfies $V(F(x)) = V(x)$ for all $x \in (\mathbb{C}^n, 0)$. The diffeomorphism $F(x)$ is *analytic integrable* in $(\mathbb{C}^n, 0)$ if it has $n - 1$ functionally independent analytic first integrals. We should mention that the notion of integrable diffeomorphisms appeared only in recent years, see for instance [9, 10, 11]. As we know, there is no a notion of integrability on diffeomorphisms defined in the broad sense as that extended by Bogoyavlenski [4] in 1998 for vector fields.

Denote by $M_n(\mathbb{C})$ the set of square matrices of order n with entries in \mathbb{C} . Let $B \in M_n(\mathbb{C})$ and $\mu = (\mu_1, \dots, \mu_n)$ be the eigenvalues of B . Recall that

- A diffeomorphism or a formal series $F(x) = Bx + f(x)$ is in *normal form* if B is in the Jordan normal form and the nonlinear term $f(x)$ consists of only resonant monomials. A monomial $x^m e_j$ in the j th component of $f(x)$ is *resonant* if $\mu^m = \mu_j$, where e_j is the unit vector with its j th component equal to 1 and others vanishing.
- A diffeomorphism (or a formal series) $G(y)$ is a *normal form* (or a *formal normal form*) of a diffeomorphism $F(x)$ if $G(y)$ is in normal form and $F(x)$ and $G(y)$ are conjugate, i.e., there is a transformation tangent to identity $y = \Phi(x) = x + \phi(x)$ with $\phi(x)$ containing only higher order terms such that $G \circ \Phi(x) = \Phi \circ F(x)$. The conjugacy $y = \Phi(x)$ is called a *normalization* from $F(x)$ to $G(y)$. Furthermore
 - $y = \Phi(x)$ is an *analytic normalization* of $F(x)$ if $\Phi(x)$ is analytic. In this case we call $G(y)$ an *analytically equivalent normal form* of $F(x)$.
 - $y = \Phi(x)$ is a *distinguished normalization* of $F(x)$ if $\phi(x)$ contains only nonresonant term, i.e. its monomial $x^m e_j$ in the j th component of $\phi(x)$ are *nonresonant* in the sense that $\mu^m \neq 1$.

We remind readers the difference between the resonances of diffeomorphisms and the transformations.

Our first main result of this paper provides more information on analytic integrable diffeomorphisms than the existence of analytic normalization. Before stating the results, we introduce a notation. Let $\mu = (\mu_1, \dots, \mu_n)$ be the eigenvalues of B . Set

$$\mathcal{D} = \{m \in \mathbb{Z}_+^n; \mu^m = 1, |m| \geq 2\},$$

namely *resonant set* of B , and denote by d_μ the rank of the resonant set. The elements of \mathcal{D} are also called *resonant lattices*. An element $m \in \mathcal{D}$ is *simple* if it cannot be divided by a positive integer no less than 2.

Theorem 1.1. *For a diffeomorphism $F(x) = Bx + f(x)$ defined in $(\mathbb{C}^n, 0)$ a neighborhood of 0 in \mathbb{C}^n with $f(x) = O(|x|^2)$ and B having at least one eigenvalue not on the unit circle of \mathbb{C} , then $F(x)$ is analytic integrable if and only if the following statements hold.*

- (a) *the resonant set has the rank $d_\mu = n - 1$.*
- (b) *$F(x)$ is conjugate to its normal form of type*

$$G(y) = (\mu_1 y_1 (1 + p_1(y)), \dots, \mu_n y_n (1 + p_n(y))),$$

by a distinguished analytic normalization, where $\mu = (\mu_1, \dots, \mu_n)$ is the n -tuple of eigenvalues of B , and $p_1(y), \dots, p_n(y)$ are analytic and satisfy the equations

$$(1 + p_1(y))^{m_{k1}} \dots (1 + p_n(y))^{m_{kn}} = 1, \quad \text{for } k = 1, \dots, n - 1,$$

with $m_k = (m_{k1}, \dots, m_{kn}) \in \mathcal{D}$, $k = 1, \dots, n - 1$, being $n - 1$ linearly independent simple resonant lattices.

We remark that this last result is a correction and improvement of the one given in [32]. From this last theorem and the following Lemma 2.5(c) we can get easily the following.

Corollary 1.2. *For analytic integrable diffeomorphism $F(x) = Bx + f(x)$ with $f(x)$ nonlinear, if the orbits of the normal form system of $F(x)$ and of the linear one $\mu y = (\mu_1 y_1, \dots, \mu_n y_n)$ start at the same generic point, then the full orbits will be contained in the same orbit of an analytic vector field, where $\mu = (\mu_1, \dots, \mu_n)$ is the n -tuple of eigenvalues of B and not all of μ_i 's on the unit circle of \mathbb{C} .*

Recall that a *generic point* is the one which is located in a full Lebesgue measure subset of $(\mathbb{C}^n, 0)$.

As we know, for higher dimensional local analytic diffeomorphisms the existence of analytic normalization is solved only for the diffeomorphisms having their linear parts with eigenvalues $\mu = (\mu_1, \dots, \mu_n)$ either all larger (or smaller) than 1 in modulus, or satisfying $|\mu^m - \mu_s| \geq c|m|^{-\nu}$ for all $s = 1, \dots, n$, $|m| \geq 2$, with $c, \nu > 0$ given constants. The former result is called *Poincaré–Dulac theorem*, and the latter is called *Siegel theorem*, see e.g. §25 of [1]. We note that in the Siegel theorem the eigenvalues are nonresonant. In our case the eigenvalues of the linear parts of the diffeomorphisms can be resonant, and their modulus can have part of them larger than 1 and have also other part of them less than 1.

We now turn to the study of the problem on the existence of analytic normalizations for analytic integrable vector fields to their normal forms. We will see that this problem is simpler than that for analytic integrable diffeomorphisms.

Consider the analytic differential system

$$\dot{x} = Ax + f(x), \quad x \in (\mathbb{C}^n, 0), \quad (1.1)$$

where $A \in M_n(\mathbb{C})$, and $f(x) = O(|x|^2)$ is a vector-valued analytic function in $(\mathbb{C}^n, 0)$. We say that system (1.1) is *locally complete analytic integrable* in $(\mathbb{C}^n, 0)$ if it has $n-1$ functionally independent analytic first integrals in $(\mathbb{C}^n, 0)$. An *analytic first integral* of system (1.1) is a nonconstant analytic function $H(x)$ defined in $(\mathbb{C}^n, 0)$ satisfying $\langle \nabla H(x), Ax + f(x) \rangle = 0$ in $(\mathbb{C}^n, 0)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors in \mathbb{C}^n , and ∇ represents the gradient of a function with respect to x . The $n-1$ analytic first integrals are *functionally independent* if their gradients as vectors in \mathbb{C}^n are linearly independent on an open dense subset of $(\mathbb{C}^n, 0)$.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the n -tuple of eigenvalues of the matrix A . Set

$$\mathcal{R}_\lambda := \{m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n; \langle m, \lambda \rangle = 0, |m| = m_1 + \dots + m_n \geq 2\},$$

where \mathbb{Z}_+ denotes the set of nonnegative integers. We call \mathcal{R}_λ *resonant set* of λ , and its elements *resonant lattices*. Denote by r_λ the rank of vectors in the set \mathcal{R}_λ . Obviously, if $\lambda \neq 0$ then $r_\lambda \leq n-1$. In what follows we assume without loss of generality that A is in the lower triangular Jordan normal form.

In [32] we have proved the following result. *Assume that the origin of system (1.1) is nondegenerate, i.e. no eigenvalues equal to zero, and that the matrix A is diagonalizable. Then system (1.1) has $n-1$ locally functionally independent analytic first integrals if and only if $r_\lambda = n-1$, and system (1.1) is analytically equivalent to its distinguished normal form $\dot{y}_i = \lambda_i y_i(1 + g(y))$, $i = 1, \dots, n$, by an analytic normalization, where $g(y)$, without constant term, is an analytic function of y^m with $m \in \mathcal{R}_\lambda$ and $(m_1, \dots, m_n) = 1$, i.e. they have no common factor. Recall by definition that $y^m = y_1^{m_1} \dots y_n^{m_n}$ for $y = (y_1, \dots, y_n)$ and $m = (m_1, \dots, m_n)$.*

We say that system (1.1) is in *normal form* if A is in Jordan normal form and the Taylor series of $f(x)$ consists of only resonant monomials. A monomial $x^m e_j$ modulo coefficient in the j th component of $f(x)$ is *resonant* if $\lambda_j = \langle m, \lambda \rangle$, where $m \in \mathbb{Z}_+^n$, $|m| \geq 2$.

A system

$$\dot{y} = Ay + g(y), \quad (1.2)$$

with $g(y)$ containing only higher order terms is a *normal form* of (1.1) if system (1.2) is in normal form, and there is a change of variables tangent to identity $y = \Psi(x)$ transforming system (1.1) to (1.2). The transformation $y = \Psi(x)$ is called a *normalization*. If the transformation contains only nonresonant terms, then it is called *distinguished normalization*. Correspondingly, the normal form is called a *distinguished normal form*. Recall that a monomial $x^m e_j$ in the transformation is *nonresonant* if $\langle m, \lambda \rangle \neq 0$.

Related to the above results we posed in Remark 2 of [32] the following open problem: *if system (1.1) has the origin as a degenerate singularity, and has $n - 1$ locally functionally independent analytic first integrals in a neighborhood of the origin, is system (1.1) locally analytically equivalent to its distinguished normal forms?*

In this paper we will give a positive answer to this problem. The following is our second main result of this paper.

Theorem 1.3. *Assume that $n \geq 2$ and $\lambda \neq 0$, i.e. A has at least one eigenvalues not equal to zero. Then system (1.1) has $n - 1$ functionally independent analytic first integrals in $(\mathbb{C}^n, 0)$ if and only if the rank of the resonant set is equal to 1, i.e. $r_\lambda = n - 1$, and system (1.1) is analytically equivalent to its distinguished normal form*

$$\dot{y}_i = \lambda_i y_i (1 + g(y)), \quad i = 1, \dots, n, \quad (1.3)$$

by an analytic normalization tangent to the identity, where $g(y)$, without constant term, is an analytic function of y^m with $m \in \mathcal{R}_\lambda$ and $(m_1, \dots, m_n) = 1$.

This last theorem characterizes the complete analytic integrable differential systems. It provides not only the existence of analytic normalization for analytic integrable differential systems but also the concrete expression of their normal forms. Theorem 1.3 also implies that the linear part of an analytic integrable differential system must be diagonalizable provided that it has at least one nonzero eigenvalue. We can improve the corresponding result given in [32], because we find that in the degenerate case there also does not appear small divisors in the normalization.

We should mention that one dimensional differential equation, if it is nontrivial, has no nonconstant first integrals. For higher dimensional systems, if all eigenvalues λ of A are equal to zero, Theorem 1.3 cannot be applied, see the following examples.

Example 1. The planar system

$$\dot{x} = y + f(x, y), \quad \dot{y} = 0,$$

with $f(x, y)$ an analytic function without linear and constant terms, has the analytic first integral $H(x, y) = y$. Also the planar analytic system

$$\dot{x} = y^{p+1}, \quad \dot{y} = -x^{q+1}, \quad p, q \in \mathbb{N},$$

has the analytic first integral $H(x, y) = x^{q+2}/(q+2) + y^{p+2}/(p+2)$. But these two analytic integrable systems cannot be transformed to systems of form (1.3) by an invertible analytic change of coordinates, because the linear part of these two systems have only zero eigenvalues. Also we note that according to the definition of resonance the nonlinear parts of these last two systems are all resonant.

These last examples show that the condition $\lambda \neq 0$ is necessary for Theorem 1.3. So we have the following

Open problem 1. Assume that the linear part of system (1.1) has all eigenvalues vanishing.

- What is the normal form that an analytic integrable system (1.1) can have?
- Is an analytic integrable system (1.1) analytically equivalent to its normal form?

Comparing with Theorem 1.3, we must mention the work of Zung [38], in which author studied the existence of analytic normalization for obtaining the Poincaré–Dulac normal form of an analytic integrable differential systems in the broad sense. In 1998 Bogoyavlenski [4] extended the classical complete integrable differential systems including the Liouvillian integrable Hamiltonian systems as follows: a local analytic (or smooth) vector field in $(\mathbb{C}^n, 0)$ is *analytic (or smooth) integrable in the broad sense* if for some natural number q ($1 \leq q \leq n$), there exist q locally analytic (or smooth) vector fields $X_1 = X, X_2, \dots, X_q$ and $n - q$ locally analytic (or smooth) functions f_1, \dots, f_{n-q} in $(\mathbb{C}^n, 0)$ such that

- i)* the q vector fields X_1, \dots, X_q commute pairwise and are linearly independent almost everywhere;
- ii)* f_1, \dots, f_{n-q} are common first integrals of X_1, \dots, X_q and are functionally independent almost everywhere.

Using a geometric method Zung [38] in 2002 proved that any analytic integrable differential system in the broad sense is analytically conjugate to its normal form. From the existence of analytic normalization point of view, Zung's result contains our Theorem 1.3 as a special

case. But our result can present the concrete expression of the normal form of the system. Since we do not study this broad sense integrability, in the following when we say integrable systems, we always mean the complete integrable systems which were defined above.

We note that the distinguished normal form of an analytic integrable vector field has a concrete expression, but we cannot present the exact expression of their normal forms for analytic integrable diffeomorphisms. In fact, they depend on the resonant lattices. This can be seen from the following concrete examples, which illustrate some applications of Theorem 1.1.

Example 2. According to Theorem 1.1, the two dimensional analytic integrable diffeomorphism $(\frac{1}{2}x + g_1(x, y), 2y + g_2(x, y))$ is locally analytically conjugate to its distinguished normal form

$$F_1(x, y) = \left(\frac{1}{2}x \left(1 - \frac{\varphi(xy)}{1 + \varphi(xy)} \right), 2y(1 + \varphi(xy)) \right),$$

with $\varphi(z)$ an analytic function satisfying $\varphi(0) = 0$. $F_1(x, y)$ has the same first integral $H(x, y) = xy$ as the linear diffeomorphism $L_1(x, y) = (\frac{1}{2}x, 2y)$. But they cannot be parallel as in the case of two dimensional vector fields.

The three dimensional analytic integrable diffeomorphism

$$(e^{-5}x + g_1(x, y, z), e^2y + g_2(x, y, z), ez + g_3(x, y, z)),$$

is locally analytically conjugate to its normal form

$$F_2(x, y, z) = \left(e^{-5}x \frac{1}{(1 + \psi)^{5/2}}, e^2y(1 + \psi), ez(1 + \psi)^{1/2} \right),$$

which together with the linear diffeomorphism $L_2(x, y, z) = (e^{-5}x, e^2y, ez)$ have the functionally independent analytic first integrals $H_1(x, y, z) = x^2y^5$ and $H_2(x, y, z) = xy^2z$, where $\psi = \psi(w_1, w_2, w_3, w_4) = \psi(xy^2z, xyz^3, xz^5, x^2y^5)$ is an analytic function in its variables and $\psi(0, 0, 0, 0) = 0$. We note that xyz^3 and xz^5 are also analytic first integrals of F_2 and of L_2 , and they functionally depend on xy^2z and x^2y^5 . But they cannot be represented in analytic functions of xy^2z and x^2y^5 .

In addition, any three dimensional locally analytic integrable diffeomorphism

$$G(x, y, z) = (x + g_1(x, y, z), y + g_2(x, y, z), 2z + g_3(x, y, z)),$$

with g_1, g_2, g_3 nonlinear, is analytically conjugate to $(x, y, 2z(1 + h(x, y)))$ with h an analytic function in x and y .

The last three examples show that the normal forms of higher dimensional analytic integral diffeomorphisms have more complicated expressions than those of vector fields.

Now we briefly review the results on the existence of analytic normalizations for analytic integrable vector fields to their normal forms and also on the existence of analytic integrable vector fields.

The following result, known as Poincaré normal form theorem of a nondegenerate center, goes back to Poincaré and Lyapunov (see e.g. [28]): a planar analytic differential system has the origin as a nondegenerate center if and only if it is analytically equivalent (via probably complex transformation of variables and time rescaling) to

$$\dot{x} = x(1 + q(xy)), \quad \dot{y} = -y(1 + q(xy)), \quad (1.4)$$

where $q(u)$ is an analytic function in u starting from the terms of degree no less than 1. We note that this result is a special case of our Theorem 1.3, because in the case of nondegenerate center the unique linearly independent simple resonant lattices is $(1, 1)$ under the complex coordinates. The Poincaré normal form theorem has a corollary as follows: a planar analytic differential system has the origin as an isochronous center if and only if it is analytically equivalent to

$$\dot{u} = -\omega v, \quad \dot{v} = \omega u,$$

where ω is a nonzero constant. Moser [24] showed that a planar real analytic Hamiltonian system having the origin as a hyperbolic saddle can be reduced to system (1.4) by a real analytic area-preserving transformation of variables. This shows that Theorem 1.3 is not only a generalization of the above Poincaré's and Moser's results to higher dimensional systems in nondegenerate cases, but also a generalization of their results to degenerate cases. For example, the planar analytic differential system $\dot{x} = 0$, $\dot{y} = \lambda y + o(|x, y|^2)$ with $\lambda \neq 0$ is analytically equivalent to a system of form $\dot{x} = 0$, $\dot{y} = y(\lambda + O(x))$ by Theorem 1.3, because the system has a functionally independent analytic first integral. For general planar analytic differential systems, Llibre *et al.* [6, 7, 15] characterized their locally analytic integrability around a singularity with the aid of normal forms. Closely related to analytic integrability of planar differential systems, the existence of inverse analytic integrating factors provides much more information on the dynamics of the system (see e.g. [12, 13, 14]).

Our study on the existence of analytic normalization and the concrete expressions of the normal forms for complete analytic integrable systems in \mathbb{C}^n is also motivated by the study on a similar problem for Hamiltonian systems, that is, on the existence of analytically symplectic normalizations which transform analytic integrable symplectic Hamiltonian systems to their Birkhoff normal form. Ito [18, 19] solved this problem under the restriction that the eigenvalues of linear part of systems are nonresonant and simple resonant, respectively. Zung [37] completely solved this problem, and proved that analytic Liouvillian integrable

symplectic Hamiltonian system is analytically symplectically equivalent to its Birkhoff normal form by developing a new geometric method based on the toric characterization of Birkhoff normalization. Recently Ito [20] presented a relation between superintegrability of Hamiltonian systems and the existence of analytic Birkhoff normalization.

Here we mainly concern the analytic normalization for analytic integrable systems. For general differential systems including Hamiltonian ones, there are extensive studies on the existence of analytic normalizations, we refer readers to the papers [31, 30], the books [22, 17] and the references therein. On the generic nonexistence of analytic normalization for analytic differential systems, we refer readers to Siegel [29] and Pérez-Marco [26, 27].

Theorems 1.1 and 1.3 show that both analytic integrable diffeomorphisms and analytic integrable vector fields have analytic normalizations. It motivates to think whether an analytic integrable diffeomorphism can be embedded in an analytic integrable autonomous vector field.

The following result was given in [32] (for a similar one, see Cima *et al* [10]). *Any analytic integrable volume-preserving diffeomorphism defined on an analytic manifold \mathcal{M} can be embedded in an analytic flow on \mathcal{M} .*

Here we will release the restriction on volume-preserving of the diffeomorphisms and will give a global proof on the given manifold where the integrable diffeomorphisms are defined. Our result is the following

Theorem 1.4. *Let \mathcal{M} be a real or complex n -dimensional analytic manifold. Then any real or complex analytic integrable diffeomorphism defined on \mathcal{M} can be embedded in an analytic flow on \mathcal{M} .*

Using the same method as that in the proof of Theorem 1.4, we can get easily the following result.

Corollary 1.5. *Let \mathcal{M} be an n -dimensional C^k smooth manifold with $k \in \{\mathbb{N}\} \cup \{\infty\}$. Then any C^k smoothness integrable diffeomorphism defined on \mathcal{M} can be embedded in a C^{k-1} smoothness flow on \mathcal{M} , where $\infty - 1 = \infty$.*

These last results have solved the open problem given in Remark 5 of [32]. Recall that for an analytic or smooth manifold \mathcal{M} , a diffeomorphism $F : \mathcal{M} \rightarrow \mathcal{M}$ can be *embedded in a flow* ϕ_t if $\phi_1 = F$ on \mathcal{M} . The vector field $\partial\phi_t/\partial t|_{t=0}$ is called *embedding vector field* of F . Generally, a time dependent vector field $\mathcal{X}(t, x)$ is an *embedding vector field* of the diffeomorphism $F(x)$ if the latter coincides with the time 1 map of solutions of $\mathcal{X}(t, x)$.

In one dimensional case there are rich results on the embedding flow problem (see e.g. [2, 22] and the references therein). In higher dimensional cases, Arnold [1] posed the

following result without a proof that if $A \in M_n(\mathbb{R})$ has a real logarithm B , i.e. $A = e^B$, then any C^∞ local diffeomorphism $f(x) = Ax + O(|x|^2)$ can be embedded in a C^∞ periodic vector field $\dot{x} = Ax + g(t, x)$ in $(\mathbb{R}^n, 0)$, where $g(t+1, x) = g(t, x)$ and $g(t, x) = O(|x|^2)$, for a proof see [23, Lemma 16]). Kuksin and Pöschel [21] proved the existence of C^∞ or analytic periodic embedding Hamiltonian vector fields for a class of nearly integrable C^∞ or analytic symplectic diffeomorphisms defined in $(\mathbb{R}^{2m}, 0)$.

The problem on the existence of embedding flows or embedding autonomous vector fields becomes more difficult, as mentioned by Arnold in [1, p.200]. Palis [25] proved that the diffeomorphisms admitting embedding flows are rare in the Baire sense. In [23], we proved that for a C^∞ diffeomorphisms $F(x) = Ax + f(x)$ defined in $(\mathbb{R}^n, 0)$ with A having a real logarithm B and $f(x) = O(|x|^2)$, if A has no eigenvalues on the unit circle of \mathbb{C} and the eigenvalues of B are not weakly resonant, then $F(x)$ can be embedded in a C^∞ autonomous vector field. This result was recently extended to Banach spaces [33]. For analytic diffeomorphisms, as we know, Theorem 1.3 of [32] and part of Theorem 1.4 of [34] are the only results on the existence of analytic embedding flows in higher dimensional spaces. Recently Zhang [35] provided a simple proof to the result of [23] mentioned above and presented some examples showing that the weakly nonresonant conditions of the real logarithm B of A is necessary.

Theorem 1.4 has solved the embedding flow problem for analytic integrable diffeomorphisms. But for nonintegrable analytic diffeomorphisms the problem is still open.

Open problem 2. To characterize all analytic diffeomorphisms which admit analytic embedding flows.

The paper is organized as follows. We first prove Theorem 1.1 in Section 2. Because the proof of Theorem 1.3 is similar to that of Theorem 1.1 and is easier, which will be given in Section 3, where we mainly concern the difference with that of Theorem 1.1. The last section presents the proof of Theorem 1.4.

2 Proof of Theorem 1.1

We separate the proof of Theorem 1.1 into several lemmas. One of the main tools for proving the theorem is the normal form theory. For doing so, we need an auxiliary result, which will be used later on in different ways for several times.

Lemma 2.1. *Let $\mathcal{H}_n^r(\mathbb{C})$ be the linear space of n -dimensional vector-valued homogeneous polynomials of degree r in n variables with coefficients in \mathbb{C} . For $B, C \in M_n(\mathbb{C})$, we define*

a linear operator $\mathcal{L}_{B,C}$ on $\mathcal{H}_n^r(\mathbb{C})$ by

$$(\mathcal{L}_{B,C}\phi)(x) = \phi(Bx) - C\phi(x), \quad \phi \in \mathcal{H}_n^r(\mathbb{C}).$$

Then the spectrum, denoted by $\sigma(\mathcal{L}_{B,C})$, of $\mathcal{L}_{B,C}$ is

$$\sigma(\mathcal{L}_{B,C}) = \{\mu^m - \kappa_j; m \in \mathbb{Z}_+^n, |m| = r, j = 1, \dots, n\},$$

where $\mu = (\mu_1, \dots, \mu_n)$ and $\kappa = (\kappa_1, \dots, \kappa_n)$ are the n -tuples of eigenvalues of B and C , respectively.

Proof. The idea of the proof follows from that of Lemma 1.1 of [3] and of Lemma 4.5 of [22]. Let $T, S \in M_n(\mathbb{C})$ be invertible. Set $\phi(x) = T\xi(x)$ and $x = Sy$. Then for $\psi(y) = \xi(Sy)$ we have

$$\begin{aligned} (\mathcal{L}_{B,C}\phi)(x) &= T\xi(Bx) - CT\xi(x) = T(\xi(BSy) - T^{-1}CT\xi(Sy)) \\ &= T(\psi(S^{-1}BSy) - T^{-1}CT\psi(y)). \end{aligned}$$

Consider the linear operator

$$(\mathcal{L}^*\psi)(y) = \psi(S^{-1}BSy) - T^{-1}CT\psi(y). \quad (2.1)$$

Then the linear operators $\mathcal{L}_{B,C}$ and \mathcal{L}^* have the same spectrum, because $\psi(y) = T^{-1}\phi(Sy)$, and S and T are invertible. So without loss of generality we can assume that the matrices B and C are in lower triangular Jordan normal form.

Case 1. B and C are diagonalizable. From (2.1), we can assume without loss of generality that B and C are diagonal. For any monomial $h(x) = x^m e_j \in \mathcal{B} = \{x^m e_j; m \in \mathbb{Z}_+^n, |m| = r, j = 1, \dots, n\}$ a base of $\mathcal{H}_n^r(\mathbb{C})$, where e_j is the j th unit vector, we have

$$\mathcal{L}_{B,C}(x^m e_j) = (\mu x)^m e_j - \text{diag}(\kappa_1, \dots, \kappa_n)x^m e_j = (\mu^m - \kappa_j)x^m e_j.$$

This shows that the matrix expression of the linear operator $\mathcal{L}_{B,C}$ under the base \mathcal{B} is diagonal with $\mu^m - \kappa_j$ being the elements on the diagonal entries. Hence $\mathcal{L}_{B,C}$ has the spectrum as stated in the lemma.

Case 2. At least one of B and C is not diagonalizable. Choose $B(\varepsilon), C(\varepsilon) \in M_n(\mathbb{C})$ such that $B(\varepsilon) \rightarrow B$ and $C(\varepsilon) \rightarrow C$ as $\varepsilon \rightarrow 0$ and that $B(\varepsilon)$ and $C(\varepsilon)$ are both diagonalizable. Let $\mu(\varepsilon) = (\mu_1(\varepsilon), \dots, \mu_n(\varepsilon))$ and $\kappa(\varepsilon) = (\kappa_1(\varepsilon), \dots, \kappa_n(\varepsilon))$ be the n -tuples of eigenvalues of the matrices $B(\varepsilon)$ and $C(\varepsilon)$, respectively. Then it follows from the proof of Case 1 that the linear operator $\mathcal{L}_{B(\varepsilon), C(\varepsilon)}$ has the spectrum $\sigma(\mathcal{L}_{B(\varepsilon), C(\varepsilon)}) = \{\mu(\varepsilon)^m - \kappa_j(\varepsilon); m \in \mathbb{Z}_+^n, |m| = r, j = 1, \dots, n\}$. Since the linear operator $\mathcal{L}_{B,C}$ depends on B and C continuously, and $\mu(\varepsilon) \rightarrow \mu$ and $\kappa(\varepsilon) \rightarrow \kappa$ as $\varepsilon \rightarrow 0$, we get the spectrum of $\mathcal{L}_{B,C}$ as stated in the lemma. The proof is completed.

The first result is on the existence of formal normal form for analytic diffeomorphisms, which can be found in any book when it introduces normal form theory (see e.g. [1, 3, 22]).

Lemma 2.2. *The analytic diffeomorphism $F(x) = Bx + f(x)$ is always formally conjugate to its distinguished normal form by a formal transformation tangent to the identity.*

Proof. We present its proof here because which will be used in the proof of our other results. For the diffeomorphism $F(x) = Bx + f(x)$, we can assume without loss of generality that B is in lower triangle Jordan normal form. Because there always exists an invertible linear conjugation Cy such that $F(x)$ is conjugate to $C^{-1}BCy + C^{-1}h(Cy)$.

Suppose that $F(x) = Bx + f(x)$ is conjugated to $G(y) = By + g(y)$ via a formal conjugation tangent to the identity, i.e. it is of the form $x = \Phi(y) = y + \phi(y)$ with $\phi(y)$ a formal series starting from at least the second order term. Then we get from $F \circ \Phi(y) = \Phi \circ G(y)$ that g and ϕ satisfy

$$\phi(By) - B\phi(y) = f(y + \phi(y)) + \phi(By) - \phi(By + g(y)) - g(y). \quad (2.2)$$

Expanding $h \in \{f, g, \phi\}$ in Taylor series gives

$$h(x) = \sum_{i=l}^{\infty} h_i(x),$$

where h_i is a vector-valued homogeneous polynomial of degree i , and l is the degree of the lowest order homogeneous polynomial in the Taylor expansion of $f(x)$. Then we get from (2.2) that

$$\phi_s(By) - B\phi_s(y) = [f]_s + [\phi]_s - g_s(y), \quad s = l, l+1, \dots, \quad (2.3)$$

where $[f]_s$ and $[\phi]_s$ are inductively known vector-valued homogeneous polynomials in y of degree s obtained by re-expanding $f(y + \phi(y))$ and $\phi(By) - \phi(By + g(y))$ in Taylor series in y , respectively. In fact, we have $[\phi]_l = 0$.

Recall that $\mathcal{H}_n^s(\mathbb{C})$ is the linear space formed by n -dimensional vector-valued complex homogeneous polynomials of degree r in n variables. Define a linear operator $\mathcal{L}_{B,s} : \mathcal{H}_n^s(\mathbb{C}) \rightarrow \mathcal{H}_n^s(\mathbb{C})$ by

$$\mathcal{L}_{B,s}\phi(y) = \phi(By) - B\phi(y) \quad \text{for } \phi \in \mathcal{H}_n^s(\mathbb{C}).$$

We get from Lemma 2.1 that the spectrum of $\mathcal{L}_{B,s}$ is

$$\left\{ \prod_{i=1}^n \mu_i^{m_i} - \mu_j; m_i \in \mathbb{Z}_+, \sum_{i=1}^n m_i = s, j = 1, \dots, n \right\},$$

where $(\mu_1, \dots, \mu_n) = \mu$ is the n -tuple of eigenvalues of B .

For each $s \in \mathbb{N}$, we separate $\mathcal{H}_n^s(\mathbb{C})$ into two parts, i.e. $\mathcal{H}_n^s(\mathbb{C}) = \mathcal{H}_n^{sr} \oplus \mathcal{H}_n^{sn}$, where \mathcal{H}_n^{sr} (resp. \mathcal{H}_n^{sn}) consists of vector-valued resonant (resp. nonresonant) homogeneous polynomials of degree s . Correspondingly, we separate the right hand side of (2.3) into $[f]_s + [\phi]_s - g_s = ([f]_{sr} + [\phi]_{sr} - g_{sr}) + ([f]_{sn} + [\phi]_{sn} - g_{sn}) \in \mathcal{H}_n^{sr} \oplus \mathcal{H}_n^{sn}$. Since the operator $\mathcal{L}_{B,s}$ is linear, equation (2.3) can be written in two equations

$$\mathcal{L}_{B,s}\phi_{sr} = [f]_{sr} + [\phi]_{sr} - g_{sr}, \quad (2.4)$$

$$\mathcal{L}_{B,s}\phi_{sn} = [f]_{sn} + [\phi]_{sn} - g_{sn}, \quad (2.5)$$

with $\phi_s = \phi_{sr} + \phi_{sn} \in \mathcal{H}_n^{sr} \oplus \mathcal{H}_n^{sn}$. For equation (2.4), we choose $g_{sr} = [f]_{sr} + [\phi]_{sr}$, and consequently it has the trivial solution $\phi_{sr} = 0$. Since $\mathcal{L}_{B,s}$ is invertible on \mathcal{H}_n^{sn} , equation (2.5) has always a unique solution ϕ_{sn} for any $g_{sn} \in \mathcal{H}_n^{sn}$. We choose the solution of (2.5) with $g_{sn} = 0$. Then $\phi_s(y) = \phi_{sn}(y)$ contains only nonresonant monomials, and $g_s(y) = g_{sr}$ contains only resonant monomials.

The above proof shows that the normalization $\Phi(y) = y + \phi(y)$ has its nonlinear part consisting of nonresonant terms, i.e. all monomials $y^m e_i$ in the i th component satisfying $\mu^m - \mu_i \neq 0$. The distinguished normal form $G(y) = By + g(y)$ has its nonlinear part consisting of resonant terms. Moreover, we know from the above proof that the distinguished normal form and normalization are both unique. This proves the lemma.

The next result characterizes the first integral of the distinguished normal form for the given diffeomorphism $F(x)$.

Lemma 2.3. *Let $G(y) = By + g(y)$ be the distinguished normal form of $F(x) = Bx + f(x)$ via the distinguished normalization $x = \Phi(y) = y + \phi(y)$. If $F(x)$ has an analytic first integral, then $G(y)$ has a first integral either analytic or formal with its nonlinear term all resonant.*

We should mention the difference on resonance between first integrals and normal forms including normalization. By definition a monomial y^m in a first integral of $G(y)$ is *resonant* if $\mu^m = 1$.

Proof. By the assumption F and G are conjugate, i.e. $F \circ \Phi = \Phi \circ G$. If $V(x)$ is an analytic first integral of $F(x)$, then $W(y) = V \circ \Phi(y)$ is an analytic or formal first integral of $G(y)$ because $W(G(y)) = W \circ G(y) = V \circ \Phi \circ G(y) = V \circ F \circ \Phi(y) = V \circ \Phi(y) = W(y)$, where we have used the fact that $V(F(x)) = V(x)$ for all $x \in (\mathbb{C}^n, 0)$.

Next we prove that the first integral W of G consists of resonant monomials, i.e. its each monomial y^m modulo coefficient satisfying $\mu^m = 1$. Indeed, W is a first integral of $G(y)$ means that $W(G(y)) = W(y)$ for all $y \in (\mathbb{C}^n, 0)$. We rewrite this last equation as

$$W(By) - W(y) = W(By) - W(By + g(y)). \quad (2.6)$$

From the above proof we can set $W(y) = \sum_{s=r}^{\infty} W_s(y)$ with $W_s(y)$ homogeneous polynomial of degree s in y and $W_r(y) \not\equiv 0$. Re-expanding $W(By + g(y))$ in Taylor series in y , we get from (2.6) that

$$W_s(By) - W_s(y) = R_s(y), \quad s = r, r+1, \dots, \quad (2.7)$$

where $R_s(y)$ is inductively known and $R_r(y) = 0$. From Lemma 2.1 we get that the linear operator $\mathcal{L}_s : H^s(\mathbb{C}^n) \rightarrow H^s(\mathbb{C}^n)$ defined by $\mathcal{L}_s \phi(y) = \phi(By) - \phi(y)$ for $\phi \in H^s(\mathbb{C}^n)$ has the spectrum $\{\mu^m - 1; m \in \mathbb{Z}_+^n, |m| \geq 2\}$, where $H^s(\mathbb{C}^n)$ is the linear space of scalar complex homogeneous polynomials of degree s in n variables. So equation (2.7) with $s = r$, i.e. $\mathcal{L}_r W_r(y) = 0$, has only the resonant homogeneous polynomial solution. So W_r must be a resonant homogeneous polynomial of degree r . For equation (2.7) with $s > r$, it is easy to know from the right hand side of (2.6) that the $R_s(y)$ is constructed from W_r, \dots, W_{s-1} . By induction we assume that W_r, \dots, W_{s-1} are all resonant. So in order for proving $R_s(y)$ to be resonant, we only need to prove that each term z^m in $W(By + g(y))$ with $m \in \mathbb{Z}_+^n$, $|m| < s$ and $z = By + g(y)$ contains only resonant monomials in y . For this aim, we set B in lower triangular Jordan normal form with

$$B = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ \sigma_1 & \mu_2 & 0 & 0 & 0 \\ 0 & \sigma_2 & \ddots & 0 & 0 \\ 0 & 0 & \sigma_{n-2} & \mu_{n-1} & 0 \\ 0 & 0 & 0 & \sigma_{n-1} & \mu_n \end{pmatrix},$$

where $\sigma_i = 0$ or 1 for $i = 1, \dots, n-1$, and if $\sigma_i = 1$ then $\mu_{i-1} = \mu_i$. In these notations we have $z^m = \prod_{i=1}^n (\sigma_{i-1} y_{i-1} + \mu_i y_i + g_i(y))^{m_i}$, where $\sigma_0 = 0$. Using the binormal expansion

we get $(\sigma_{i-1} y_{i-1} + \mu_i y_i + g_i(y))^{m_i} = \sum_{j=0}^{m_i} \binom{m_i}{j} (\sigma_{i-1} y_{i-1} + \mu_i y_i)^{m_i-j} (g_i(y))^j$. Since $g_i(y)$

contains only resonant monomials, i.e. its each monomial y^k satisfies $\mu^k = \mu_i$, it follows that each monomial y^l in $(g_i(y))^j$ satisfies $\mu^l = \mu_i^j$. In addition, if $\sigma_{i-1} \neq 0$ then y_{i-1} and y_i satisfy the same resonant conditions because the eigenvalues corresponding to y_{i-1} and y_i are the same. These two facts show that the monomial y^r in $(\sigma_{i-1} y_{i-1} + \mu_i y_i)^{m_i-j} (g_i(y))^j$ verifies $\mu^r = \mu_i^{m_i}$. Consequently each monomial y^q in z^m satisfies $\mu^q = \mu_1^{m_1} \dots \mu_n^{m_n} = 1$, where we have used the fact that each monomial z^m of $W_i(z)$ for $l \leq i < s$ is resonant. This proves the lemma.

We now study the number of functionally independent analytic or formal first integrals for analytic diffeomorphisms.

Lemma 2.4. *The analytic diffeomorphism $F(x) = Bx + f(x)$ has at most d_μ , the rank of resonant set \mathcal{D} of B , functionally independent analytic or formal first integrals.*

Proof. Let $G(y) = By + g(y)$ be the distinguished normal form of $F(x)$ through a normalization tangent to the identity, and let $W(y)$ be an analytic or a formal first integral of G . It follows from Lemma 2.3 that $W(y)$ is also a first integral of the linear map μy with $\mu y = (\mu_1 y_1, \dots, \mu_n y_n)$, because $W(y)$ consists of resonant monomials and each monomial y^m in W satisfies $(\mu y)^m = \mu^m y^m = y^m$.

Since $F(x)$ and $G(y)$ are conjugate via an analytic or a formal normalization tangent to the identity, they have the same number of functionally independent analytic or formal first integrals. So we only need to prove that $G(y)$ has at most d_μ functionally independent analytic or formal first integrals. The above proof shows that the the number of functionally independent first integrals of $G(y)$ does not exceed that of μy . We now turn to prove that the linear diffeomorphism μy has exactly d_μ functionally independent analytic first integrals.

Let m_1, \dots, m_{d_μ} be the linearly independent elements of \mathcal{D} , and each vector m_j for $j = 1, \dots, d_\mu$ is simple, i.e. it cannot be divided by a positive integer no less than 2. Then $y^{m_1}, \dots, y^{m_{d_\mu}}$ are the d_μ functionally independent analytic first integrals of μy , because $(\mu y)^{m_k} = y^{m_k}$ for $k = 1, \dots, d_\mu$ and their gradients $\nabla(y^{m_k}) = ((m_{k1}/y_1, \dots, m_{kn}/y_n)y^{m_k}$, $k = 1, \dots, d_\mu$, are linearly independent in an open dense subset of \mathbb{C}^n . For any $m^* \in \mathcal{D}$, it linearly depends on m_1, \dots, m_{d_μ} , and so y^{m^*} functionally depends on $y^{m_1}, \dots, y^{m_{d_\mu}}$. Also the proof of Lemma 2.3 shows that any formal or analytic first integral of μy consists of resonant monomials. This implies that any analytic or formal first integral of μy is an analytic or a formal function of y^m with $m \in \mathcal{D}$ simple. We have proved that μy has exactly d_μ functionally independent analytic or formal first integrals. Consequently $F(x)$ has at most d_μ functionally independent analytic or formal first integrals. The proof is completed.

We now study the expression of the normal form $G(y)$ and the relation between the orbits of $G(y) = By + g(y)$ and of μy .

Lemma 2.5. *Assume that the analytic diffeomorphism $F(x) = Bx + f(x)$ has $n - 1$ functionally independent analytic first integrals, and that $G(y) = By + g(y)$ is the distinguished normal form of $F(x)$. If B has at least one eigenvalue not equal to one in modulus, then the following statements hold.*

- (a) *The resonant set of B has the rank $d_\mu = n - 1$.*
- (b) *B is diagonal, and $G(y) = (\mu_1 y_1(1 + p_1(y)), \mu_2 y_2(1 + p_2(y)), \dots, \mu_n y_n(1 + p_n(y)))$ with*

$p_i(0) = 0$ for $i = 1, \dots, n$. Moreover, $p_1(y), \dots, p_n(y)$ satisfy the functional equations

$$(1 + p_1(y))^{m_{k1}} \dots (1 + p_n(y))^{m_{kn}} = 1, \quad k = 1, \dots, n-1,$$

where $(m_{k1}, \dots, m_{kn}) = m_k \in \mathcal{D}$, $k = 1, \dots, n-1$, are linearly independent and simple.

- (c) The generic orbits of both $G(y)$ and μy are contained in the same orbits of some vector field.

Proof. (a) Since the eigenvalues μ have modulus not all equal to 1, it implies that the rank d_μ of the resonant set of B is less than or equal to $n-1$. By the assumption and Lemma 2.4 we get that $d_\mu = n-1$.

(b) Let $V_1(x), \dots, V_{n-1}(x)$ be the $n-1$ functionally independent analytic first integrals without constants of $F(x)$. Then it follows from the proof of Lemma 2.3 that $W_i(y) = V_i \circ \Phi(y)$, $i = 1, \dots, n-1$, are the functionally independent analytic or formal first integrals of $G(y)$, where $x = \Phi(y)$ is the distinguished normalization from $F(x)$ to $G(y)$. By Ziglin's lemma [36] (see also the appendix of [18]), we can assume without loss of generality that the lowest order parts $W_i^0(y)$ of $W_i(y)$ for $i = 1, \dots, n-1$ are functionally independent. Otherwise it can be done by polynomial combination of these integrals with complex coefficients.

Since $W_i(y)$ consists of resonant monomials, we get that $W_i(By + g(y)) = W_i(y)$ and $W_i(\mu y) = W_i(y)$ for all y in $(\mathbb{C}^n, 0)$. Equating the lowest order terms in y of these last two equations gives $W_i^0(By) = W_i^0(y) = W_i^0(\mu y)$. Set $B = U + N$ with $U = \text{diag}(\mu_1, \dots, \mu_n)$ and N in the nilpotent lower triangular normal form, i.e. we have $Ny = (0, \sigma_1 y_1, \dots, \sigma_{n-1} y_{n-1})$. Then we get from $W_i^0(\mu y + Ny) = W_i^0(\mu y)$ that

$$\langle \nabla W_i^0(\mu y + \theta_y Ny), Ny \rangle = 0, \quad i = 1, \dots, n-1, \quad (2.8)$$

where $\theta_y \in (0, 1)$ and $\mu y = Uy$. Since W_1^0, \dots, W_{n-1}^0 are functionally independent in $(\mathbb{C}^n, 0)$, we can assume without loss of generality that

$$\Delta^*(y) = \det \begin{pmatrix} \frac{\partial W_1^0}{\partial x_2}(z_y) & \dots & \frac{\partial W_1^0}{\partial x_n}(z_y) \\ \vdots & \ddots & \vdots \\ \frac{\partial W_{n-1}^0}{\partial x_2}(z_y) & \dots & \frac{\partial W_{n-1}^0}{\partial x_n}(z_y) \end{pmatrix} \neq 0,$$

in an open subset of $(\mathbb{C}^n, 0)$, where $z_y = \mu y + \theta_y Ny$. Otherwise it can be done by rearranging the order of the coordinates, and meanwhile the Jordan normal form B keeps in the same form. Hence equation (2.8) has the unique solution $Ny = 0$, and consequently $N = 0$. This proves that B is diagonal.

For proving $G(y)$ to have the special type of normal form, instead of the $n-1$ functionally independent first integrals $W_i(y)$ we consider the $n-1$ functionally independent monomial first integrals $H_k(y) = y^{m_k}$ for $k = 1, \dots, n-1$ with $m_1, \dots, m_{n-1} \in \mathcal{D}$ being linearly independent and simple. Here the existence of the $n-1$ monomial first integrals $H_k(y)$ follows from the facts that since $W_1(y), \dots, W_{n-1}(y)$ are functional independent, and so the Inverse Function Theorem implies that there exist $n-1$ functionally independent monomials $H_1(y), \dots, H_{n-1}(y)$ in one-to-one way on an open and dense subset such that they are first integrals of $G(y)$, i.e. $H_i(G(y)) = H_i(y)$ hold in an open and dense subset of $(\mathbb{C}^n, 0)$ for $i = 1, \dots, n-1$. Now $H_i(y)$ are monomials and $G(y)$ is an analytic function or a formal series force that $H_i(G(y)) = H_i(y)$ must hold in $(\mathbb{C}^n, 0)$, because by expanding these last equations and equating the homogeneous terms of the same order, we get a series of homogeneous polynomial equations. They hold in an open and dense subset of $(\mathbb{C}^n, 0)$ and so must hold in $(\mathbb{C}^n, 0)$.

Since B is diagonal, we have $G(y) = (\mu_1 y_1 + g_1(y), \dots, \mu_n y_n + g_n(y))$. In the next proof we distinguish two cases: either for any $k \in \{1, \dots, n-1\}$, $H_k(y)$ does not contain y_1 ; or there exists some $k_0 \in \{1, \dots, n-1\}$ for which $H_{k_0}(y)$ contains y_1 .

In the former, the first components of m_k for $k = 1, \dots, n-1$ are all zero. Hence we have $\langle \overline{m}_k, \overline{\mu} \rangle = 0$ for $k = 1, \dots, n-1$, where $\overline{m}_k = (m_{k2}, \dots, m_{kn})$ and $\overline{\mu} = (\mu_2, \dots, \mu_n)$. Obviously, $\overline{m}_1, \dots, \overline{m}_{n-1}$ are linearly independent. In $n-1$ dimensional case, $\overline{\mu}$ satisfies $n-1$ linearly independent resonant relations, it follows from the proof of Lemma 2.4 that $\overline{\mu}$ has its components all having modulus 1. By the assumption we must have $|\mu_1| \neq 1$. Since $G(y)$ is in normal form, any nonlinear monomial y^k in the first component of $G(y)$ satisfies $\mu_1 = \mu^k$. So we have $|\mu_1| = |\mu_1|^{k_1}$, i.e. $k_1 = 1$. This proves that y_1 divides $g_1(y)$.

In the latter, set $g_1(y) = y_1 p_1(y) + q_1(y)$, where $p_1(y) = O(|y|)$, and $q_1(y) = O(|y|^2)$ is independent of y_1 . Using the fact that $H_{k_0}(y)$ are the first integrals of both $G(y)$ and μy , i.e. $H_{k_0}(G(y)) = H_{k_0}(y) = H_{k_0}(\mu y)$, we obtain that

$$\begin{aligned} & (\mu_1 y_1 + y_1 p_1(y) + q_1(y))^{m_{k_0 1}} (\mu_2 y_2 + g_2(y))^{m_{k_0 2}} \dots (\mu_n y_n + g_n(y))^{m_{k_0 n}} \\ &= (\mu_1 y_1)^{m_{k_0 1}} (\mu_2 y_2)^{m_{k_0 2}} \dots (\mu_n y_n)^{m_{k_0 n}}. \end{aligned}$$

In this last equation by setting $y_1 = 0$ gives

$$(q_1(y))^{m_{k_0 1}} (\mu_2 y_2 + g_2(0, y_2, \dots, y_n))^{m_{k_0 2}} \dots (\mu_n y_n + g_n(0, y_2, \dots, y_n))^{m_{k_0 n}} \equiv 0.$$

This verifies that $q_1(y) \equiv 0$, because $m_{k_0 1} \neq 0$ and $\mu_i \neq 0$ for $i = 2, \dots, n$.

The proof of the above two cases shows that the first component of the normal form $G(y)$ is of the form $\mu_1 y_1(1 + p_1(y))$. Working out in the same line we can prove that the j th component of $G(y)$ is of the form $\mu_j y_j(1 + p_j(y))$ for $j = 2, \dots, n$.

Finally using the first integrals $H_k(y)$ of $G(y)$ and of μy , we obtain that

$$(\mu_1 y_1 (1 + p_1(y)))^{m_{k1}} \dots (\mu_n y_n (1 + p_n(y)))^{m_{kn}} = (\mu y)^{m_k}, \quad k = 1, \dots, n-1.$$

Simplifying these last equations yields

$$(1 + p_1(y))^{m_{k1}} \dots (1 + p_n(y))^{m_{kn}} = 1, \quad k = 1, \dots, n-1,$$

This proves statement (b).

(c) We will use the notations given in the proof of statement (b). The above proof shows that $W_1(y), \dots, W_{n-1}(y)$ are functionally independent first integrals of both $G(y)$ and μy . So the level surfaces of W_i for $i = 1, \dots, n-1$ are invariant under the action of either $G(y)$ or μy . This implies that each orbit of $G(y)$ and of μy is contained in the level surfaces of W_i for $i = 1, \dots, n-1$ and so in their intersection. Clearly the intersection is one dimensional in the full Lebesgue measure subset of $(\mathbb{C}^n, 0)$ because of the functional independence of the $n-1$ first integrals.

Define a vector field in $(\mathbb{C}^n, 0)$ by

$$\mathcal{Z}(y) = \nabla W_1(y) \times \dots \times \nabla W_{n-1}(y), \quad \text{for } y \in (\mathbb{C}^n, 0),$$

where \times denotes the cross product of vectors in \mathbb{C}^n . Recall that the cross product of $n-1$ vectors in \mathbb{C}^n , saying v_1, \dots, v_{n-1} , is again a vector, and is defined by

$$\langle v_1 \times \dots \times v_{n-1}, w \rangle = \det \begin{pmatrix} w \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix},$$

for arbitrary $w \in \mathbb{C}^n$. By the very definition of the cross product, it is easy to check that W_k for $k = 1, \dots, n-1$ are first integrals of the vector field $\mathcal{Z}(y)$. So the orbits of the vector field $\mathcal{Z}(y)$ are contained in the intersections of the level surfaces of $W_1(y), \dots, W_{n-1}(y)$. This proves that both orbits of the diffeomorphisms $G(y)$ and μy starting at the same generic point are contained in the same orbit of $\mathcal{Z}(y)$. Recall that the generic points are those ones which are located in a full Lebesgue measure subset of $(\mathbb{C}^n, 0)$. We finish the proof of the lemma.

Next we prove that the nonresonant spectrum of a linear operator related to B is bounded from below in modulus.

Lemma 2.6. *Assume that the diffeomorphism $F(x) = Bx + f(x)$ has $n-1$ functionally independent analytic first integrals and that B has at least one eigenvalue with modulus not equal to 1. Then there exists a $\sigma > 0$ such that if $\mu^m - \mu_i \neq 0$ for $m \in \mathbb{Z}_+^n$, $|m| \geq 2$ and $i = 1, \dots, n$, we have $|\mu^m - \mu_i| \geq \sigma$.*

Proof. By the assumption of the lemma we get from Lemma 2.5 that there exist $n - 1$ linearly independent vectors $k_i = (k_{i1}, \dots, k_{in}) \in \mathbb{Z}_+^n$ such that $\mu^{k_i} = 1$ for $i = 1, \dots, n - 1$. This follows that

$$k_{i1} \log |\mu_1| + \dots + k_{in} \log |\mu_n| = 0, \quad i = 1, \dots, n - 1. \quad (2.9)$$

Since k_1, \dots, k_{n-1} are linearly independent, we can assume without loss of generality that

$$\Delta = \det \begin{pmatrix} k_{11} & \cdots & k_{1,n-1} \\ \vdots & \ddots & \vdots \\ k_{n-1,1} & \cdots & k_{n-1,n-1} \end{pmatrix} \neq 0.$$

Then we get from equation (2.9) using the Cram's rule that

$$\log |\mu_j| = \frac{\delta_j}{\Delta} \log |\mu_n|, \quad j = 1, \dots, n - 1, \quad (2.10)$$

where δ_j 's, $\Delta \in \mathbb{Z}$. Moreover we have $\log |\mu_n| \neq 0$. Otherwise all μ_j have modulus 1, a contradiction with the assumption of the lemma. Using (2.10) we get that for any $m \in \mathbb{Z}_+^n$ and $j = 1, \dots, n$

$$\begin{aligned} |\mu^m - \mu_j| &\geq ||\mu_1|^{m_1} \dots |\mu_n|^{m_n} - |\mu_j|| \\ &= |\mu_n|^{\frac{\delta_j}{\Delta}} \left| |\mu_n|^{\frac{m_1 \delta_1 + \dots + (m_j - 1) \delta_j + \dots + m_n \delta_n}{\Delta}} - 1 \right|, \end{aligned}$$

where $\delta_n = \Delta$. Set $\alpha = |\mu_n|^{1/\Delta}$. Since $s_m = m_1 \delta_1 + \dots + (m_j - 1) \delta_j + \dots + m_n \delta_n \in \mathbb{Z}$, we have either $\alpha^{s_m} - 1 = 0$ if $s_m = 0$, or $|\alpha^{s_m} - 1| \geq \min\{|\alpha - 1|, |\alpha^{-1} - 1|\} \neq 0$ if $s_m \neq 0$.

For $s_m \neq 0$, set

$$\sigma_1 = \min\{\alpha^{\delta_j} |\alpha - 1|, \alpha^{\delta_j} |\alpha^{-1} - 1|; j = 1, \dots, n\},$$

we have $|\mu^m - \mu_j| \geq \sigma_1$ for $j = 1, \dots, n$ and all $m \in \mathbb{Z}_+^n$ with $|m| \geq 2$ such that $s_m \neq 0$.

We now consider those m such that $s_m = 0$. From $\mu^{k_j} = 1$ for $j = 1, \dots, n - 1$ we get that

$$k_{j1} \log \mu_1 + \dots + k_{jn} \log \mu_n = \log 1, \quad j = 1, \dots, n - 1, \quad (2.11)$$

where $\log 1 = 2n\pi\sqrt{-1}$, $n \in \mathbb{Z}$, and the logarithms are taken for complex numbers because the eigenvalues μ may be complex. Solving (2.11) by the Cram's rule gives

$$\log \mu_j = \frac{2n_j \rho_j \pi \sqrt{-1} + \delta_j \log \mu_n}{\Delta}, \quad j = 1, \dots, n - 1, \quad (2.12)$$

where δ_j is the same as that of (2.10), $\rho_j \in \mathbb{Z}$ is uniquely determined by k_i for $i = 1, \dots, n - 1$, and $n_j \in \mathbb{Z}$ come from the expression of $\log 1$. For the $m \in \mathbb{Z}_+^n$ such that

$s_m = 0$, we have

$$\begin{aligned}
|\mu^m - \mu_j| &= |e^{m_1 \log \mu_1} \dots e^{m_n \log \mu_n} - e^{\log \mu_j}| \\
&= \left| e^{\frac{2 \sum_{k=1}^n m_k n_k \rho_k \pi \sqrt{-1} + \sum_{k=1}^n m_k \delta_k \log \mu_n}{\Delta}} - e^{\frac{2n_j \rho_j \pi \sqrt{-1} + \delta_j \log \mu_n}{\Delta}} \right| \\
&= \left| e^{\frac{2 \sum_{k=1}^n m_k n_k \rho_k \pi \sqrt{-1}}{\Delta}} \mu_n^{\frac{\sum_{k=1}^n m_k \delta_k}{\Delta}} - e^{\frac{2n_j \rho_j \pi \sqrt{-1}}{\Delta}} \mu_n^{\frac{\delta_j}{\Delta}} \right| \\
&= |\mu_n|^{\frac{\delta_j}{\Delta}} \left| e^{\frac{2 \sum_{k=1}^n m_k n_k \rho_k \pi \sqrt{-1}}{\Delta}} - e^{\frac{2n_j \rho_j \pi \sqrt{-1}}{\Delta}} \right|,
\end{aligned}$$

where we have used the fact $s_m = 0$ in the fourth equality, i.e. $\sum_{k=1}^n m_k \delta_k = \delta_j$. Since Δ, ρ_k, ρ_j are given integers which are uniquely determined by the linearly independent vectors k_i for $i = 1, \dots, n-1$, by the periodic property of the exponential functions with respect to their pure imaginary parts, it follows that

$$e^{\frac{2 \sum_{k=1}^n m_k n_k \rho_k \pi \sqrt{-1}}{\Delta}} \quad \text{and} \quad e^{\frac{2n_j \rho_j \pi \sqrt{-1}}{\Delta}} \quad (2.13)$$

both take only finitely many values for all possible choice of m_k, n_k, n_j . Taking γ to be the minimum of the modulus of all possible differences of the two elements given in (2.13). By the assumption $|\mu^m - \mu_j| \neq 0$ we have $\gamma \neq 0$, that is, the modulus of their difference has a nonzero minimum. Set

$$\sigma_2 = \min\{\alpha^{\delta_j} \gamma; j = 1, \dots, n\}.$$

Then $\sigma = \min\{\sigma_1, \sigma_2\}$ is the data satisfying the lemma, i.e. we have $|\mu^m - \mu_j| \geq \sigma$ for all $m \in \mathbb{Z}_+^n$ with $|m| \geq 2$, $j = 1, \dots, n$ and $|\mu^m - \mu_j| \neq 0$. We complete the proof of the lemma.

The last step is to prove that the normalization from $F(x)$ to $G(y)$ is convergent. Lemma 2.6 shows that in the analytic integrable case there does not appear small divisor conditions. A folklore says that if no small divisor conditions appear, it is convergent that the distinguished normalization tangent to identity from a given analytic vector field or analytic diffeomorphism to its distinguished normal form. In fact, it is not the case. See the following example, the planar analytic vector field $\dot{x} = x + \varphi(x, y)$, $\dot{y} = -y + \psi(x, y)$ is always formally equivalent to $\dot{x} = xf(xy)$, $\dot{y} = -yg(xy)$. The eigenvalues $\lambda_1 = 1, \lambda_2 = -1$ of its linear part do not satisfy small divisor conditions, because $0 \neq |q_1 \lambda_1 + q_2 \lambda_2 - 1| \geq 1$ for $q_1, q_2 \in \mathbb{Z}_+$. But generally no results guarantee the convergence of the normalization except for $f(xy) = g(xy)$ (see example 2.3 and the remark following Theorem 2.4 of [31]).

The following result shows that for analytic integrable diffeomorphism, the distinguished normalization is analytic.

Lemma 2.7. *If the analytic diffeomorphism $F(x) = Bx + f(x)$ is analytic integrable and B has at least one eigenvalue not equal to 1 in modulus, then it is analytically conjugate to its normal form of the type*

$$G(y) = (\mu_1 y_1(1 + p_1(y)), \dots, \mu_n y_n(1 + p_n(y))),$$

where $p_1(y), \dots, p_n(y)$ can be represented in analytic functions of a single analytic function, and $p_1(0) = \dots = p_n(0) = 0$.

Proof. Lemma 2.5 has showed that $F(x)$ has the prescribed normal form, and that there exist $n - 1$ linearly independent simple resonant lattice $m_k \in \mathcal{D}$, $k = 1, \dots, n - 1$, such that

$$(1 + p_1(y))^{m_{k1}} \dots (1 + p_n(y))^{m_{kn}} = 1, \quad k = 1, \dots, n - 1. \quad (2.14)$$

Since m_1, \dots, m_{n-1} are linearly independent, solving (2.14) yields that there exists an $\iota \in \{1, \dots, n\}$ such that $1 + p_1(y), \dots, 1 + p_{\iota-1}(y), 1 + p_{\iota+1}(y), \dots, 1 + p_n(y)$ can be represented in functions of $1 + p_\iota(y)$. More precisely, for $j \in \{1, \dots, \iota - 1, \iota + 1, \dots, n\}$ we have $1 + p_j(y) = (1 + p_\iota(y))^{p_j/q}$ with $p_j, q \in \mathbb{Z}$ uniquely determined by m_k 's. Obviously, p_j is an analytic function in p_ι if $|p_\iota| < 1$. Of course, if $p_\iota(y)$ is locally analytic in $(\mathbb{C}^n, 0)$, then $p_1(y), \dots, p_{\iota-1}(y), p_{\iota+1}(y), \dots, p_n(y)$ will be locally analytic in $(\mathbb{C}^n, 0)$.

In what follows we assume without loss of generality that $l = 1$. From the proof of Lemma 2.2 the diffeomorphism $F(x) = Bx + f(x)$ is formally transformed to $G(y) = By + g(y)$ by a formal distinguished normalization $x = y + \phi(y)$. Set

$$f_s(x) = \sum_{m \in \mathbb{Z}_+^n, |m| \geq l} f_s^m x^m, \quad g_s(y) = \sum_{m \in \mathbb{Z}_+^n, |m| \geq l} g_s^m y^m, \quad \phi_s(y) = \sum_{m \in \mathbb{Z}_+^n, |m| \geq l} \phi_s^m y^m,$$

for $s = 1, \dots, n$, where f_s^m , g_s^m and ϕ_s^m are the coefficients of x^m and y^m respectively, l is the degree of the lowest order term of $f(y)$, and h_s is the s th component of $h \in \{f, g, \phi\}$. Then by Lemmas 2.2 and 2.5, and comparing the coefficients of y^m in the k th component of (2.2), we get that

$$(\mu^m - \mu_k) \phi_k^m = [f_k]^m - \sum_{r \in \mathbb{Z}_+^n, r \not\leq m} \phi_k^r \mu^r P_r^{m-r} - \mu_k p_k^{m-e_k}, \quad (2.15)$$

where $[f_k]^m$ is the coefficient of y^m in the expansion of $f_k(y + \phi(y))$, $r \not\leq m$ means that $r \neq m$ and $m_s - r_s \geq 0$ for $s = 1, \dots, n$, and P_r^{m-r} is the coefficient of y^{m-r} of $P_r(y) = (1 + p_1(y))^{r_1} \dots (1 + p_n(y))^{r_n}$. Here we have used the fact that $\phi_k(By + g(y)) = \phi_k(\mu_1 y_1(1 + p_1(y)), \dots, \mu_n y_n(1 + p_n(y))) = \sum_{m \in \mathbb{Z}_+^n, |m| \geq l} \phi_k^m \mu^m y^m (1 + p_1(y))^{m_1} \dots (1 + p_n(y))^{m_n}$.

For $m \in \mathbb{Z}_+^n$ such that $\mu^m = \mu_k$, we have from (2.15) that

$$\phi_k^m = 0, \quad p_k^{m-e_k} = \mu_k^{-1} [f_k]^m, \quad (2.16)$$

where we have used the fact that $\sum_{r \in \mathbb{Z}_+^m, r \not\leq m} \phi_k^r \mu^r P_r^{m-r} = 0$. Because $P_r(y)$ contains only resonant term, it follows that $\mu^{m-r} = 1$ if $P_r^{m-r} \neq 0$, and so $\mu^r = \mu^m = \mu_k$. This implies that the monomial $\phi_k^r y^r$ in $\phi_k(y)$ is resonant and so $\phi_k^r = 0$. Recall that since $F(x) = Bx + f(x)$ is a diffeomorphism, we have $\mu_k \neq 0$ for $k = 1, \dots, n$. Then we have the estimation

$$|p_k^{m-e_k}| \leq \nu | [f_k]^m |, \quad (2.17)$$

where $\nu = \max\{1/|\mu_k|; k = 1, \dots, n\}$.

For $m \in \mathbb{Z}_+^n$ such that $\mu^m \neq \mu_k$, we have from (2.15) that

$$p_k^{m-e_k} = 0, \quad \phi_k^m = \frac{1}{\mu^m - \mu_k} \left([f_k]^m - \sum_{r \in \mathbb{Z}_+^m, r \not\leq m} \phi_k^r \mu^r P_r^{m-r} \right). \quad (2.18)$$

Furthermore, by Lemma 2.6 and the fact that $\mu^{m-r} = 1$ if $P_r^{m-r} \neq 0$ we have the estimation

$$\begin{aligned} \left| \frac{1}{\mu^m - \mu_k} \sum_{r \in \mathbb{Z}_+^m, r \not\leq m} \phi_k^r \mu^r P_r^{m-r} \right| &= \sum_{r \in \mathbb{Z}_+^m, r \not\leq m} \left(1 + \frac{|\mu_k|}{|\mu^m - \mu_k|} \right) |\phi_k^r| |P_r^{m-r}| \\ &\leq \delta \sum_{r \in \mathbb{Z}_+^m, r \not\leq m} |\phi_k^r| |P_r^{m-r}|, \end{aligned}$$

where $\delta = 1 + \sigma^{-1} \max\{|\mu_k|; k = 1, \dots, n\}$. Recall that σ is the data given in Lemma 2.6. Then we have the estimation for ϕ_k^m given in (2.18)

$$|\phi_k^m| \leq \sigma^{-1} |[f_k]^m| + \delta \sum_{r \in \mathbb{Z}_+^m, r \not\leq m} |\phi_k^r| |P_r^{m-r}|. \quad (2.19)$$

Having the above estimations we can use the majorant series to prove the convergence of $\phi_s(y)$ and of $p_s(y)$ for $s = 1, \dots, n$. For a series $h_s(y) = \sum_{m \in \mathbb{Z}_+^n} h_s^m y^m$, we define $\hat{h}_s(y) = \sum_{m \in \mathbb{Z}_+^n} |h_s^m| y^m$. For two scalar series $\xi(y)$ and $\eta(y)$, we say that the latter is a majorant series of the former, denoted by $\xi(y) \preceq \eta(y)$, if $|\xi^m| \leq \eta^m$ and $\eta^m \geq 0$, where ξ^m and η^m are the coefficients of y^m in the series $\xi(y)$ and $\eta(y)$, respectively. Under this notation we have $h_s \preceq \hat{h}_s$ for the scalar series h_s . We refer the readers to [16] for more detail information on the majorant series.

Since $F(x) = Bx + f(x)$ is analytic in $(\mathbb{C}^n, 0)$, by the Cauchy inequality there exists a polydisc $\Omega_\rho = \{|x_s| < \rho; s = 1, \dots, n\}$ in which we have

$$|f_s^m| \leq M \rho^{-|m|}, \quad \text{for } s = 1, \dots, n,$$

where $M = \max_s \sup_{\partial\Omega_\rho} \{|f_s|\}$ and f_s is the s th component of f . Clearly,

$$\tilde{f}(x) = \sum_{m \in \mathbb{Z}_+^n} M \rho^{-|m|} x^m,$$

is convergent in Ω_ρ , and $\hat{f}_s(x) \preccurlyeq \tilde{f}(x)$ for $s = 1, \dots, n$. So the majorant series $\hat{f}_s(x)$ of $f_s(x)$ is convergent in Ω_ρ , and consequently is analytic in the domain.

Since $\phi_s(y)$ and $p_s(y)$ have the coefficients satisfying (2.16) and (2.18) with the estimates (2.17) and (2.19), by some calculations we get that

$$\begin{aligned} \sum_{k=1}^n (\phi_k(y) + p_k(y)) &\preccurlyeq \sum_{k=1}^n (\hat{\phi}_k(y) + \hat{p}_k(y)) \\ &\preccurlyeq n(\sigma^{-1} + \nu) \tilde{f}(y + \hat{\phi}(y)) + \delta \sum_{k=1}^n \left(\hat{\phi}_k(y(1 + \hat{p}(y))) - \hat{\phi}_k(y) \right), \end{aligned} \quad (2.20)$$

where $y(1 + \hat{p}(y)) = (y_1(1 + \hat{p}_1(y)), \dots, y_n(1 + \hat{p}_n(y)))$. For simplicity to notation we set $\gamma = n(\sigma^{-1} + \nu)$. By the very definition of $\hat{\phi}_s$ and \hat{p}_s , in order for proving the convergence of $\sum_{k=1}^n (\hat{\phi}_k(y) + \hat{p}_k(y))$ in Ω_{ρ_*} with $\rho_* \in (0, \rho)$ to be specified later on, we only need to prove it when $y_1 = \dots = y_n = z$ and $|u| \leq \rho_*$. For doing so, we set

$$U(z) = \sum_{k=1}^n (\hat{\phi}_k(y) + \hat{p}_k(y)) \Big|_{y_1 = \dots = y_n = z}.$$

In fact, in $U(z)$ we can use only $\hat{p}_1(y)$, instead of $\sum_{k=1}^n \hat{p}_k(y)$, because at the beginning of the proof of this lemma we have proved that $p_2(y), \dots, p_n(y)$ can be represented in functions of $p_1(y)$. Since the lowest order terms of $\hat{\phi}(y)$ and of $\hat{p}_k(y)$ have degree no less than 1, it follows that $U(z)$ must be divided by z . Set $U(z) = V(z)z$. We get from (2.20) that

$$V(z) \preccurlyeq \gamma z \tilde{f}_*(V(z)) + \delta ((1 + zV(z))V(z(1 + zV(z))) - V(z)), \quad (2.21)$$

where $\tilde{f}_*(V(z))$ is $\tilde{f}(z(1 + V(z)), \dots, z(1 + V(z)))$ divided by z^2 , and it is analytic as a function of V . For obtaining (2.21) we have used the facts that $\tilde{f}(y + \hat{\phi}(y)) \preccurlyeq \tilde{f}(z + W(z), \dots, z + W(z))$ and that

$$\sum_{k=1}^n \left(\hat{\phi}_k(y(1 + \hat{p}(y))) - \hat{\phi}_k(y) \right) \preccurlyeq W(z(1 + W(z))) - W(z).$$

Set

$$T(h, z) = h - \gamma z \tilde{f}_*(h) - \delta ((1 + zh)h(z(1 + zh)) - h).$$

For studying the existence of analytic solution, saying $h(z)$, of $T(h, z) = 0$, we introduce an auxiliary function

$$\Lambda(h, z) = h - \gamma z \tilde{f}_*(h) - \delta((1 + zh)h - h).$$

Obviously Λ is analytic in h and z , because \tilde{f}_* is an analytic function in h . Some easy calculations show that

$$\Lambda(0, 0) = 0, \quad \left. \frac{\partial \Lambda}{\partial h} \right|_{(h, z) = (0, 0)} = 1.$$

By the Implicit Function Theorem the equation $\Lambda(h, z) = 0$ has a unique analytic solution, denoted by $h_0(z)$, in a neighborhood of 0 in C .

Choose $\rho_1 > 0$ satisfying $\rho_1 < \min\{1, \rho\}$ for which $h_0(z)$ is analytic in $B_{\rho_1}(0) = \{z \in \mathbb{C}; |z| < \rho_1\}$ and $\|h_0\| = \sup\{|h_0(z)|; z \in B_{\rho_1}(0)\} < 1$. Then the functional equation $T(h, z) = 0$ has an analytic solution $h(z)$ defined in $B_{\rho_1/3}(0)$. Comparing (2.21) with $T(h, z)$, it follows that $h(z)$ is a majorant series of $V(z)$. Hence $V(z)$ is analytic in $B_{\rho_1/3}(0)$, and consequently $\sum_{k=1}^n (\hat{\phi}_k + \hat{p}_k)$ is analytic in the ball. This proves that the distinguished normalization from $F(x) = Bx + f(x)$ to $G(y) = By + g(y)$ is analytic, that is, $F(x)$ is analytically conjugate to its distinguished normal form. We complete the proof of the lemma.

Having the above preparations we can prove Theorem 1.1.

Proof of Theorem 1.1: Sufficiency. By the assumption of the theorem the monomials $H_k(y) = y^{m_k}$ for $k = 1, \dots, n-1$ are $n-1$ functionally independent analytic first integrals of $G(y)$, where m_1, \dots, m_{n-1} are the linearly independent resonant lattices given in Theorem 1.1. Let $x = \Phi(y)$ be the analytic conjugation tangent to the identity from $F(x)$ to $G(y)$, and let $y = \Psi(x)$ be its inverse. Then $\Psi(x)$ is analytic, tangent to the identity and satisfies $\Psi \circ F = G \circ \Psi$. On the other hand, using the conjugate condition and $H_k \circ G(y) = H_k(y)$ for $y \in (\mathbb{C}^n, 0)$ we get that $H_k \circ \Psi \circ F(x) = H_k \circ G \circ \Psi(x) = H_k \circ \Psi(x)$. This proves that $H_k \circ \Psi(x)$, $k = 1, \dots, n$, are $n-1$ analytic first integrals of $F(x)$. Furthermore, by the functional independence of H_1, \dots, H_{n-1} and $y = \Psi(x)$ tangent to identity, it follows easily that $H_1 \circ \Psi(x), \dots, H_{n-1} \circ \Psi(x)$ are functionally independent in $(\mathbb{C}^n, 0)$. This proves that $F(x)$ has $n-1$ functionally independent analytic first integrals, and consequently is analytic integrable in $(\mathbb{C}^n, 0)$.

Necessity. The proof follows from Lemmas 2.5 and 2.7. We have completed the proof of the theorem. \square

3 Proof of Theorem 1.3

We should mention that the main idea of the proof follows from that of Theorem 1.1. Here we present a sketch proof and mainly concern the parts of the proof which are different from those given in the proof of Theorem 1.1.

Sufficiency. By the assumption $R_\lambda = n - 1$, there exist $n - 1$ linearly independent vectors $m_i = (m_{i1}, \dots, m_{in}) \in \mathbb{Z}_+^n$, $i = 1, \dots, n$, such that $\langle m_i, \lambda \rangle = 0$. This implies that y^{m_i} , $i = 1, \dots, n - 1$ are $n - 1$ functionally independent analytic first integrals of (1.3). Let $y = \psi(x)$ be the analytic transformation tangent to the origin from (1.3) to (1.1) in a neighborhood of the origin. Then ψ^{m_i} , $i = 1, \dots, n - 1$, are the $n - 1$ functionally independent analytic first integrals of (1.1).

Necessity. Denote by \mathcal{X} the vector fields induced by system (1.1). Set $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_h$ with \mathcal{X}_1 and \mathcal{X}_h the linear and higher order terms, respectively. Furthermore we separate $\mathcal{X}_1 = \mathcal{X}_1^s + \mathcal{X}_1^n$ with $\mathcal{X}_1^s = \langle A_1 x, \partial_x \rangle$ the *semisimple part* and $\mathcal{X}_1^n = \langle A_2 x, \partial_x \rangle$ the *nilpotent part* of \mathcal{X}_1 respectively, where $A = A_1 + A_2$. Without loss of generality, we can assume that

$$\mathcal{X}_1^s := \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}.$$

Recall that the vector field \mathcal{X} is in *normal form* is equivalent to that the Lie bracket of \mathcal{X}_1^s and \mathcal{X}_h vanishes, i.e. $[\mathcal{X}_1^s, \mathcal{X}_h] = 0$.

For a given analytic differential system or vector field, by the Poincaré-Dulac normal form theorem it can always be transformed to its distinguished normal form by a distinguished normalization. Let

$$\dot{y} = Ay + g(y), \tag{3.1}$$

be the distinguished normal form of (1.1) obtained from the normalization $x = \Phi(y) = y + \varphi(y)$. Then the vector field associated with (3.1) is $\mathcal{Y}(y) = (D\Phi(y))^{-1}(A + f) \circ \Phi(y)$, where $D\Phi(y)$ denotes the Jacobian matrix of $\Phi(y)$.

We claim that if $H(x)$ is an analytic first integral of (1.1), then $V(y) = H(y + \varphi(y))$ is an analytic or formal first integral of (3.1), and all its monomials are resonant. The proof is similar to Lemma 2.3 of [32]. The difference is that now A is a priori not necessary diagonal. We now prove the claim. That $H(x)$ is an analytic first integral of (1.1) is equivalent to $\langle \partial_x H, Ax + F(x) \rangle = 0$. By the chain rule, it follows that $\langle \nabla V(y), \mathcal{Y}(y) \rangle = 0$, where $\nabla V(y)$ denotes the gradient of $V(y)$. This shows that $V(y)$ is a first integral (analytically or formally) of system (3.1).

Write

$$V(y) = \sum_{k=l}^{\infty} V_k(y), \quad \mathcal{Y}(y) = Ay + \sum_{j=2}^{\infty} G_j(y)$$

with $l \geq 1$ a suitable natural number and $V_k(y)$ homogeneous polynomial in y of degree k for $k = l, l+1, \dots$, and $G_j(y)$ vector-valued homogeneous polynomial in y of degree j for $j = 2, 3, \dots$. Then we get from $\langle \nabla V(y), \mathcal{Y}(y) \rangle = 0$ that

$$\langle \nabla V_l, Ay \rangle = 0, \quad (3.2)$$

$$\langle \nabla V_m, Ay \rangle = - \sum_{j=2}^m \langle \nabla V_{m+1-j}, G_j \rangle, \quad m = l+1, l+2, \dots \quad (3.3)$$

From the Bibikov's result [3], the linear operator \mathcal{L}_r from $H_r^n(y)$ to itself defined by

$$\mathcal{L}_r h = \langle \nabla h(y), Ay \rangle, \quad h(y) \in H_r^n(y)$$

has the spectrum $\sigma(\mathcal{L}_r) = \{\langle \kappa, \lambda \rangle; \kappa \in \mathbb{Z}_+^n, |\kappa| = r\}$. So, the solution V_l of equation (3.2) should consist of resonant monomials of degree l . Otherwise, it is null by the spectrum of \mathcal{L}_l .

For each $m \in \{l+1, l+2, \dots\}$, the right hand side of (3.3) is an inductively known resonant polynomials of degree m , because G_j and V_{m+1-j} are resonant homogeneous polynomials in a vector field and in a function, respectively. Hence it follows from the spectrum of the linear operator \mathcal{L}_m that V_m is a resonant homogeneous polynomial of degree m . This proves the claim.

By the assumption of the theorem, system (1.1) has $n-1$ functionally independent analytic first integrals, denoted by $H_1(x), \dots, H_{n-1}$. From the Ziglin's lemma [36] (see also the appendix of [18]), considering polynomials of these $n-1$ first integrals with complex coefficients, we may assume without loss of generality that the lowest order homogeneous polynomials of these first integrals are functionally independent.

Set $V_i(y) = H_i \circ \Phi(y)$ for $i = 1, \dots, n-1$. The last claim shows that $V_i(y)$, $i = 1, \dots, n-1$, are functionally independent first integrals of the distinguished normal form vector field \mathcal{Y} of \mathcal{X} . And each $V_i(y)$ contains only resonant terms. Moreover, the lowest order homogeneous polynomials, saying $V_i^0(y)$, of $V_i(y)$ for $i = 1, \dots, n-1$ are also functionally independent.

The first integrals $V_i(y)$ of $\mathcal{Y}(y)$ satisfy the equations $\langle \nabla V_i(y), Ay + g(y) \rangle = 0$. Equating the lowest order terms of these last equations, we get that $\langle \nabla V_i^0(y), Ay \rangle = 0$. Since $V_i^0(y)$ are composed of resonant monomials, it follows that $\langle \nabla V_i^0, \lambda y \rangle = 0$ for $i = 1, \dots, n-1$, where $\lambda y = (\lambda_1 y_1, \dots, \lambda_{n-1} y_{n-1})$. This proves that both Ay and λy are orthogonal to the $n-1$ dimensional linear space spanned by $\nabla V_1, \dots, \nabla V_{n-1}$ in an open and dense subset of $(\mathbb{C}^n, 0)$. Hence the vectors Ay and λy must be parallel in an open and dense subset of $(\mathbb{C}^n, 0)$ because we are in the n -dimensional space. This implies that $Ay = \lambda y$ holds in an open and dense subset of $(\mathbb{C}^n, 0)$ and consequently $Ay = \lambda y$ hold in $(\mathbb{C}^n, 0)$ because they

are linear in y . This proves that if system (1.1) has $n - 1$ functionally independent analytic first integrals, then the linear part A of (1.1) should be diagonalizable.

Next we will prove that the distinguished normal form of (1.1) has the form (1.3). From the above proof we can assume that system (1.1) has the distinguished normal form of the form $\mathcal{Y} = (\lambda_1 y_1 + g_1(y), \dots, \lambda_n y_n + g_n(y))$. From the assumption of the theorem and the above proof, we know that the vector field \mathcal{Y} has $n - 1$ functionally independent first integrals, and each one consists of resonant polynomials. So working in a similar way to the proof of that A is diagonal, we can verify that the two vector fields \mathcal{Y} and λy should be parallel at each point y in a neighborhood of the origin. This implies that there exists a function of the form $1 + g(y)$ such that $\mathcal{Y} = (\lambda_1 y_1(1 + g(y)), \dots, \lambda_n y_n(1 + g(y)))$.

The remainder is to prove that the distinguished normalization from the vector fields \mathcal{X} to \mathcal{Y} is analytic. For this aim we first show that if system (1.1) has $n - 1$ functionally independent analytic first integrals, then there exists a $\kappa > 0$ such that $|\langle m, \lambda \rangle - \lambda_i| > \kappa$ for all $m \in \{m \in \mathbb{Z}_+^n; \langle m, \lambda \rangle - \lambda_i \neq 0, |m| \geq 2\}$.

Indeed, by Theorem 1.1 of [8], i.e. the number of analytic first integrals of system (1.1) is less than or equal to R_λ , we get that $R_\lambda = n - 1$. So there exist $n - 1$ linearly independent vectors $k_i = (k_{i1}, \dots, k_{in}) \in \mathbb{Z}_+^n$ with $|k_i| \geq 2$, $i = 1, \dots, n - 1$ such that

$$\langle k_i, \lambda \rangle = 0, \quad i = 1, \dots, n - 1. \quad (3.4)$$

Since k_1, \dots, k_{n-1} are linearly independent, we can assume without loss of generality that

$$\det \begin{pmatrix} k_{1,1} & \dots & k_{1,n-1} \\ \vdots & & \vdots \\ k_{n-1,1} & \dots & k_{n-1,n-1} \end{pmatrix} \neq 0.$$

Solving (3.4) gives

$$\lambda_1 = \frac{\nu_1}{\mu_1} \lambda_n, \dots, \lambda_{n-1} = \frac{\nu_{n-1}}{\mu_{n-1}} \lambda_n, \quad (3.5)$$

with $\mu_i \in \mathbb{Z} \setminus \{0\}$, $\nu_i \in \mathbb{Z}$, and μ_i, ν_i relatively prime for $i = 1, \dots, n - 1$. We note that μ_i and ν_i are uniquely determined by the k_j for $j = 1, \dots, n - 1$. Since $\lambda \neq 0$, it follows that $\lambda_n \neq 0$. If $\langle m, \lambda \rangle - \lambda_i \neq 0$ for $m \in \mathbb{Z}_+^n$ and $|m| \geq 2$, it follows from (3.5) that $|\langle m, \lambda \rangle - \lambda_i| \geq |\lambda_n|/(\mu_1 \dots \mu_{n-1}) = \kappa$. This proves the claim.

This last proof shows that there does not appear the so called small divisors in the distinguished normalization from an analytic integrable system in $(\mathbb{C}^n, 0)$ to its normal form. Then working in the same way as in the proof of Lemma 2.6 of [32], we can prove that the distinguished normalization from system (1.1) to its normal form (1.3) is uniformly convergent in a neighborhood of the origin. The details are omitted. We should mention

that in our theorem part of the eigenvalues can be zero, so for getting the coefficients of $g(y)$ in the normal form vector field \mathcal{Y} from (2.6) of [32] we must choose those $s \in \{1, \dots, n\}$ for which $\lambda_s \neq 0$. We complete the proof of the theorem.

4 Proof of Theorem 1.4

The proof of the theorem is an improvement of that given in [32] for the proof of Theorem C. Let F be the analytic integrable diffeomorphism defined on the n -dimensional analytic manifold \mathcal{M} . By the assumption, the diffeomorphism F has $n - 1$ functionally independent analytic first integrals, denoted by V_1, \dots, V_{n-1} .

Let $\{U_\alpha\}$ be coordinate charts of \mathcal{M} with $\bigcup U_\alpha = \mathcal{M}$, and x be the coordinate on U_α . Then each level surface $V_i(x) = c_i$ is invariant under the action of $F(x)$ because by definition we have $V_i(F(x)) = V_i(x)$ for all $x \in U_\alpha$. This indicates that each orbit of $F(x)$ is contained in $\bigcap_{i=1}^n \{x \in \mathcal{M}; V_i(x) = c_i\} := \gamma_c$ for some $c = (c_1, \dots, c_n) \in \mathbb{F}^n$, with either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$.

For each $y \in U_\alpha \subset \mathcal{M}$, since F is a diffeomorphism on \mathcal{M} , there exists some $x \in U_\beta \subset \mathcal{M}$ (β may be α or not) such that $y = F(x)$. Define a vector field on \mathcal{M} by

$$\mathcal{X}(y) = \det(DF(x))(\nabla V_1(F(x)) \times \dots \times \nabla V_{n-1}(F(x))), \quad \text{for } y \in U_\alpha,$$

where D denotes the Jacobian matrix of F with respect to x , $\nabla V_i(F(x)) = \nabla V_i(y)|_{y=F(x)}$ for $i = 1, \dots, n - 1$, and \times denotes the cross product of vectors in \mathbb{F}^n . We mention that the cross product of $n - 1$ vectors in \mathbb{F}^n is defined in the proof of Lemma 2.5, and that for $v_1, \dots, v_{n-1} \in \mathbb{F}^n$ their cross product $v = v_1 \times \dots \times v_{n-1}$ is orthogonal to each v_i for $i = 1, \dots, n - 1$. More generally, for $y = F^k(x)$ with some $x \in \mathcal{M}$ we have

$$\mathcal{X}(y) = \det((DF^k)(x))(\nabla V_1(F^k(x)) \times \dots \times \nabla V_{n-1}(F^k(x))),$$

where $(DF^k)(x) = (DF)(F^{k-1}(x))(DF^{k-1})(x)$.

By the very definition of γ_c and of $\mathcal{X}(y)$, it follows that $\mathcal{X}(y)$ is an analytic vector field and is tangent to each γ_c at $y \in \gamma_c$. So in order for proving \mathcal{X} to be an embedding vector field of $F(y)$, we only need to prove $DF(y)\mathcal{X}(y) = \mathcal{X} \circ F(y)$ for all $y \in \mathcal{M}$. Because for the flow $\phi_t(y)$ of $\mathcal{X}(y)$ we have $D\phi_t(y)\mathcal{X}(y) = \mathcal{X} \circ \phi_t(y)$.

For any $y = F(x) \in \mathcal{M}$, since $V_i(F(y)) = V_i(y)$ for $i = 1, \dots, n - 1$, we have

$$\nabla V_i(F(y))DF(y) = \nabla V_i(y). \tag{4.1}$$

It follows from the definition of $\mathcal{X}(y)$ and (4.1) that

$$\begin{aligned} DF(y)\mathcal{X}(y) &= \det(DF(x))DF(F(x)) \\ &\quad (\nabla V_1(F^2(x))(DF)(F(x)) \times \dots \times \nabla V_{n-1}(F^2(x))(DF)(F(x))). \end{aligned} \quad (4.2)$$

In addition, for $z = F(y) = F^2(x)$ and any vector $w(z) \in T_z\mathcal{M}$ the tangent space of \mathcal{M} at z , it follows from the definition of cross product that

$$\begin{aligned} &\langle w(F^2(x)), DF(F(x)) (\nabla V_1(F^2(x))DF(F(x)) \times \dots \times \nabla V_{n-1}(F^2(x))DF(F(x))) \rangle \\ &= \langle w(F^2(x))DF(F(x)), \nabla V_1(F^2(x))DF(F(x)) \times \dots \times \nabla V_{n-1}(F^2(x))DF(F(x)) \rangle \\ &= \det \begin{pmatrix} w(F^2(x))DF(F(x)) \\ \nabla V_1(F^2(x))DF(F(x)) \\ \vdots \\ \nabla V_{n-1}(F^2(x))DF(F(x)) \end{pmatrix} = \det(DF(F(x))) \det \begin{pmatrix} w(F^2(x)) \\ \nabla V_1(F^2(x)) \\ \vdots \\ \nabla V_{n-1}(F^2(x)) \end{pmatrix}. \end{aligned}$$

This shows that

$$\begin{aligned} &DF(F(x)) (\nabla V_1(F^2(x))DF(F(x)) \times \dots \times \nabla V_{n-1}(F^2(x))DF(F(x))) \\ &= \det(DF(F(x))) (\nabla V_1(F^2(x)) \times \dots \times \nabla V_{n-1}(F^2(x))). \end{aligned} \quad (4.3)$$

Combining (4.2) and (4.3) we get that for $y = F(x)$

$$\begin{aligned} DF(y)\mathcal{X}(y) &= \det(DF(x)) \det(DF(F(x))) (\nabla V_1(F^2(x)) \times \dots \times \nabla V_{n-1}(F^2(x))) \\ &= \det(DF^2(x)) (\nabla V_1(F^2(x)) \times \dots \times \nabla V_{n-1}(F^2(x))) \\ &= \mathcal{X} \circ F(y). \end{aligned}$$

This shows that $\mathcal{X}(y)$ is an embedding vector field of $F(y)$. We complete the proof of the theorem.

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